LINEAR ALGEBRA LECTURES

LECTURE 2

- WHAT WAS COVERED IN LECTURE 1?

– LINEAR SYSTEMS CAN BE DESCRIBED BY AN AUGMENTED MATRIX

$$\begin{aligned} 2x+5y&=7 \\ 4x-8y&=14 \end{aligned} \Longrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 2 & 5 & 7 \\ 4 & -8 & 14 \end{array}\right] \begin{array}{l} \leftarrow \text{eqn 1} \\ \leftarrow \text{eqn 2} \end{array} \\ \begin{array}{ccc} x & y & \# \end{array} \end{array}$$

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– WE CAN SOLVE THESE SYSTEMS WITH GAUSS-JORDAN ELIMINATION.

$$\left[\begin{array}{cc|c} 2 & 5 & 7 \\ 4 & -8 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 5/2 & 7/2 \\ 4 & -8 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 5/2 & 7/2 \\ 0 & -18 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 5/2 & 7/2 \\ 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & 0 \end{array}\right]$$ which means $x=7/2$, $y=0$ is the solution.

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– ADDING AND SUBTRACTING MATRICES

$A=\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 2 \end{bmatrix}$ $B=\begin{bmatrix} 3 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$ need to be the same size and shape.

$$A+B=\begin{bmatrix} 2+3 & 1+3 & 3+2 \\ 5+4 & 4+6 & 2-1 \end{bmatrix}=\begin{bmatrix} 5 & 4 & 5 \\ 9 & 10 & 1 \end{bmatrix}$$ ~~A B=~~

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$$A-B=\begin{bmatrix} 2-3 & 1-3 & 3-2 \\ 5-4 & 4-6 & 2+1 \end{bmatrix}=\begin{bmatrix} -1 & -2 & 1 \\ 1 & -2 & 3 \end{bmatrix}$$

– VECTORS

a vector is a matrix of either only one column or one row

Note: the arrow on top means "vector" (ex. $\vec{x}$)

$$\vec{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{y}=\begin{bmatrix} 0 & -1 & 2 & 4 & 7 \end{bmatrix}$$

other ways to look at vectors

$$\vec{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}= 1\hat{x}+2\hat{y}+3\hat{z}$$ ($\hat{x}$ is the unit vector in the x-direction) — means of length one

$(\hat{i}=\hat{x},\ \hat{j}=\hat{y},\ \hat{k}=\hat{z})$

$$=1\hat{i}+2\hat{j}+3\hat{k}$$

$$=1\vec{e}_1+2\vec{e}_2+3\vec{e}_3$$

$\left(\vec{e}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ e_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ldots\ e_k=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{kth row}\right)$

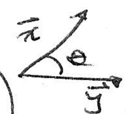
WHAT WAS COVERED IN LECTURE 1? (Continued)

- VECTOR MULTIPLICATION: DOT PRODUCT

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \vec{x} \circ \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_i x_i y_i$$

define a norm (length) as : $\vec{x} \circ \vec{x} = \|\vec{x}\|^2$ ← known as the L_2 norm

so, $\vec{x} \circ \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos \theta$ $\quad$ θ is the angle between $\vec{x}$ and $\vec{y}$



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(NOTE: if $\vec{x} \circ \vec{y} = 0$, then $\vec{x} = \vec{0}$, $\vec{y} = \vec{0}$ or $\vec{x}$ is perpendicular to $\vec{y}$)

- VECTOR MULTIPLICATION : CROSS PRODUCT

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \vec{x} \times \vec{y} = \text{determinant} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$\vec{x} \times \vec{y} = \hat{x}(x_2 y_3 - x_3 y_2) - \hat{y}(x_1 y_3 - x_3 y_1) + \hat{z}(x_1 y_2 - x_2 y_1) = \vec{b}$$

(NOTE: $\vec{b}$ will be perpendicular to $\vec{x}$ and to $\vec{y}$)

$\|\vec{x} \times \vec{y}\| = \|\vec{x}\| \, \|\vec{y}\| \sin\theta$ page 53

- MATRIX RANK
- MATRIX MULTIPLICATION
- MATRIX - VECTOR PRODUCT

- WHAT DOES "LINEAR" MEAN? IT'S AN ADJECTIVE USED TO DESCRIBE A NOUN. TYPICALLY, THE NOUN MAY BE "OPPERATOR", TRANSFORMATION", OR "RELATION".

"LINEAR" IS LIKE AN INTERFACE. ANY LINEAR OBJECT HAS TWO PROPERTIES: (SAY WE ARE TALKING ABOUT A TRANSFORMATION $T(\vec{x})$)

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for any $\vec{x}$ and $\vec{y}$ in the domain.
2. $T(c\vec{x}) = cT(\vec{x})$ where c is any ~~constant~~ scalar.

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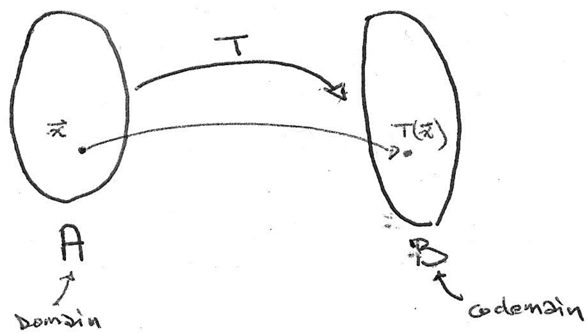
- WHAT IS A TRANSFORMATION ANYWAYS?



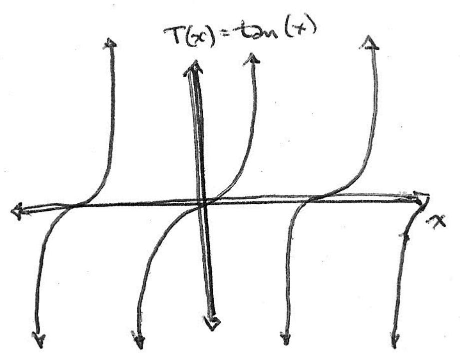
for the situation where $A = B = \mathbb{R}$, we are very familiar with a transformation.

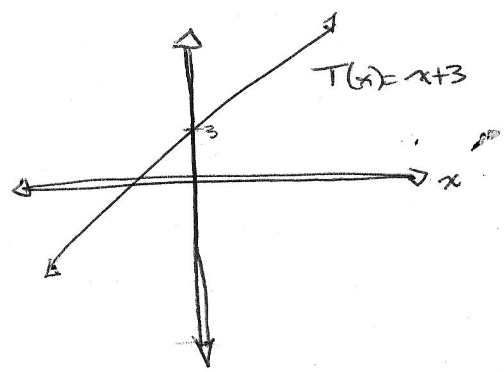
such as $T(x) = \tan(x)$

$T(x) = x+3$

$T(x) = 2^x$

WE VISUALIZE THIS ON A GRAPH THAT HAS BOTH DOMAIN AND RANGE (CODOMAIN)





EXAMPLES OF TRANSFORMATIONS: WHICH ARE LINEAR?

$T(x) = \tan(x)$

$T(x) = x+3$

$T(x) = 5x$

$T(x) = 2$

$T(x) = 0$

$T(x) = \ln(x)$

$T(x) = x^2$

$T(x) = \cos(x)$

$T(x) = x$

$T(x) = \int_0^x e^{-t^2}\,dt$

$T(x,y) = xy$

$T(x,y) = 5x + 2y$

$T(x,y) = 12y + 3 + 15x$

$T(x,y) = 15y$

$T(t) = (5t, 3t)$ ← parametric.

- IS IT POSSIBLE TO REPRESENT <u>ALL</u> LINEAR TRANSFORMATIONS WITH A MATRIX?

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suppose we have a linear transformation $T(\vec{x})$ $\quad T: \mathbb{R}^n \to \mathbb{R}^k$

we want to find a matrix M such that $M\vec{x}$ is the same as $T(\vec{x})$.

M can be characterized by ~~row~~ COLUMN vectors

$$\cancel{M = \begin{bmatrix} -\vec{m}_1- \\ -\vec{m}_2- \\ \vdots \\ -\vec{m}_k- \end{bmatrix}} \qquad M = \begin{pmatrix} | & | & & | \\ \vec{m}_1 & \vec{m}_2 & \cdots & \vec{m}_n \\ | & | & & | \end{pmatrix}$$

we need ~~$M(\vec{x})$~~ $M\vec{x} = T(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^n$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$$

since T is linear, $T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n)$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots + x_n T(\vec{e}_n) = \sum_i x_i T(\vec{e}_i)$$

now $M\vec{x} = \begin{pmatrix} | & | & & | \\ \vec{m}_1 & \vec{m}_2 & \cdots & \vec{m}_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_i x_i \vec{m}_i$

so, if $M\vec{x} = T(\vec{x})$, then $\sum_i x_i \vec{m}_i = \sum_i x_i T(\vec{e}_i) \quad \forall \vec{x} \in \mathbb{R}^n$

$$\Rightarrow \vec{m}_i = T(\vec{e}_i)$$

$$\Rightarrow \boxed{M = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}}$$